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New effective coupled $F({}^{(4)}R, \varphi)$ modified gravity from $f({}^{(5)}R)$ gravity in five dimensions

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Abstract Using some ideas of the Wesson induced matter theory, we obtain a new kind of $F({}^{(4)}R, \varphi)$ modified gravity theory as an effective four-dimensional (4D) theory derived from $f({}^{(5)}R)$ gravity in five dimensions (5D). This new theory exhibits a different matter coupling than the one in BBHL theory. We show that the field equations of the Wesson induced matter theory and of some brane-world scenarios can be obtained as maximally symmetric solutions of the same $f({}^{(5)}R)$ theory. We found criteria for the Dolgov–Kawasaki instabilities for both the $f({}^{(5)}R)$ and the $F({}^{(4)}R, \varphi)$ theories. We demonstrate that under certain conditions imposed on the 5D geometry it is possible to interpret the $F({}^{(4)}R, \varphi)$ theory as a modified gravity theory with dynamical coefficients, making this new theory a viable candidate to address the present accelerating cosmic expansion issue. Matter sources in the $F({}^{(4)}R, \varphi)$ case appear induced by the 5D geometry without the necessity of the introduction of matter sources in 5D.

1 Introduction

The acceleration in the expansion of the universe observed since 1998 by the Supernova Cosmology Project [1], has generated a great quantity of research in gravitation. In the quest of a satisfactory explanation of these phenomena, theoretical physicists have basically followed three lines of reasoning [2]. First they search for some new properties of standard gravity models capable to bring about an explanation. Second, they attribute the acceleration to a dark energy component of the universe. However, in view of the problems arising from this idea, related mainly with the nature and origin of dark energy, many other researchers resort to a third class of theories: modified gravity theories. We can find in the literature a plenty of proposals such as scalar-tensor theories [3,4], $f(R)$ theories [5,6], DGP gravity [7], brane-world scenarios [8–10], induced matter theory [11–13] and modified gravity with dynamical coefficients [14], among many others.

During the last decade, $f(R)$ theories have receive a great deal of attention because represent a possibility to address the cosmic accelerating expansion and dark matter issues [15]. In order to have a viable $f(R)$ theory there is a minimal criterio, for instance, the theory must reproduce the cosmic dynamics in good agreement with observations and the theory must be free from instabilities. One very common in the matter sector is the Dolgov–Kawasaki instability [16]. The consideration of physically different instabilities yields remarkably similar stability conditions [17,18]. Ghosts instabilities may also be present [19].

With the idea to have viable modified theories of gravity, generalizations of $f(R)$ theories have been proposed [20]. One example of non-minimal $f(R)$ gravity theories are those which exhibe couplings of the scalar curvature with matter, like the Bertolami, Bohmer, Harko, and Lobo (BBHL) theory [21,22]. In these kind of models a fifth force on massive particles appears causing changes in the acceleration law derived in the weak field limit of BBHL theory, in a similar manner than in the acceleration law in MOND models [23].

In this letter, using some ideas of the induced matter theory, we derive a new effective coupled $F({}^{(4)}R, \varphi)$ modified theory of gravity, from a five-dimensional $f({}^{(5)}R)$ theory, where the fifth extra coordinate is considered to be extended (non-compact). Our main interest is to obtain a 4D $F({}^{(4)}R, \varphi)$ theory in which matter sources are induced by the 5D geometry, in a similar manner than they are obtained in the Wesson's induced matter theory, instead of entering them a priori. Thus our interest lies on 5D vacuum solutions, even when for generality we will introduce a 5D energy-momentum tensor.

The letter is organized as follows. In Sect. 1 we give a brief introduction. In Sect. 2 we obtain the 5D field equations of the theory, together with a formulation of a criterion to avoid the Dolgov–Kawasaki instability in 5D. In adition,

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we show how the field equations of some brane-world models and of the induced matter theory can be obtained as particular maximally symmetric solutions of the 5D theory. In Sect. 3 we obtain the induced effective 4D field equations of the $F({}^{(4)}R, \varphi)$ theory and discuss its Dolgov–Kawasaki instability criterion. Some examples of how to induced a $F({}^{(4)}R, \varphi)$ from its 5D analog $f({}^{(5)}R)$ are also included in this section. Finally in Sect. 4 we give some final comments. In our conventions Latin indices like a, b, c, etc. run from 0 to 4, Latin indices like i, j, etc., run from 1 to 3, and Greek indices take values from 0 to 3.

2 Dynamical aspects of $f({}^{(5)}R)$-gravity

Let us start considering a $f(R)$ theory of gravity in five dimensions described by the action

$$ {}^{(5)}\mathcal{S} = \frac{1}{2\kappa_5}\int d^5y\sqrt{g_5}\left[f({}^{(5)}R) + \mathcal{L}_m(g_{ab}, \psi)\right], \tag{1} $$

κ_5 being the 5D gravitational coupling, ${}^{(5)}R$ the 5D Ricci scalar, $\mathcal{L}_m(g_{ab}, \psi)$ a lagrangian density for matter fields denoted by ψ and g_5 the determinant of the 5D metric tensor g_{ab}. The field equations derived from the action (1) in the metric formalism read

$$ \begin{aligned} & f_{,R}({}^{(5)}R)\,{}^{(5)}R_{ab} - \frac{1}{2}f({}^{(5)}R)\,g_{ab} \\ & -[\nabla_a\nabla_b - g_{ab}\,{}^{(5)}\Box]\,f_{,R}({}^{(5)}R) = \kappa_5\,{}^{(5)}T_{ab}, \end{aligned} \tag{2} $$

where $T_{ab}(\psi)$ is the energy-momentum tensor for matter sources, ∇_a is the 5D covariant derivative, ${}^{(5)}\Box = g^{ab}\nabla_a\nabla_b$ is the 5D D'Alambertian operator and $f_{,R}$ denotes derivative with respect to ${}^{(5)}R$.

By taking the trace of Eq. (2) we obtain

$$ 4\,{}^{(5)}\Box f_{,R} + f_{,R}\,{}^{(5)}R - \frac{5}{2}f = \kappa_5\,{}^{(5)}T \tag{3} $$

with ${}^{(5)}T \equiv g^{AB}\,{}^{(5)}T_{AB}$ being the trace of ${}^{(5)}T_{ab}$. This equation will allow us to study some important aspects of the $f({}^{(5)}R)$ theory, as it is usually done in common 4D $f(R)$ theories.

Once we have the field equations of the $f({}^{(5)}R)$ theory, we are in a position to study its stability.

2.1 Dolgov–Kawasaki instability in 5D

In order to derive a Dolgov–Kawasaki instability criterion on this $f({}^{(5)}R)$ gravity theory let us use the parametrization

$$ f({}^{(5)}R) = {}^{(5)}R + \gamma\zeta({}^{(5)}R), \tag{4} $$

where γ is a small parameter with $[\text{length}]^{-2}$ units. This election of $f({}^{(5)}R)$ means that we are considering deviations of the theory from 5D general relativity. Inserting (4) in (3) and evaluating the 5D D'Alambertian we obtain

$$ \begin{aligned} & {}^{(5)}\Box\,{}^{(5)}R + \frac{\zeta^{(3)}}{\zeta^{(2)}}\nabla^a\,{}^{(5)}R\nabla_a\,{}^{(5)}R + \frac{\left(\gamma\zeta^{(1)} - \frac{5}{2}\right)\,{}^{(5)}R}{4\gamma\zeta^{(2)}} \\ & = \frac{\kappa_5\,{}^{(5)}T}{4\gamma\zeta^{(2)}} + \frac{5\zeta}{8\zeta^{(2)}}, \end{aligned} \tag{5} $$

where $\zeta^{(i)}$ denotes the higher derivative of order i with respect to ${}^{(5)}R$. We are also assuming that $\zeta^{(2)} \neq 0$ to avoid the 5D general relativity case. Following a similar procedure to that employed in [17], we will consider the weak field limit conditions

$$ g_{ab} = \eta_{ab} + H_{ab}, \quad {}^{(5)}R = -\kappa_5\,{}^{(5)}T + {}^{(5)}R_1, \tag{6} $$

where η_{ab} is the 5D Minkowsky metric, H_{ab} is a 5D metric fluctuation tensor respect to the Minkowsky background, ${}^{(5)}R_1$ is a first order perturbation to ${}^{(5)}R$, and $|{}^{(5)}R/\kappa_5\,{}^{(5)}T| \ll 1$ with ${}^{(5)}T \neq 0$. Thus, linearizing Eq. (5) it leads to

$$ \begin{aligned} & {}^{(5)}\ddot{R}_1 - \nabla^2\,{}^{(5)}R_1 - {}^{(5)}\overset{\star\star}{R}_1 - \frac{2\kappa_5\zeta^{(3)}}{\zeta^{(2)}}\,{}^{(5)}\dot{T}\,{}^{(5)}\dot{R}_1 \\ & + \frac{2\kappa_5\zeta^{(3)}}{\zeta^{(2)}}\nabla\,{}^{(5)}T\cdot\nabla\,{}^{(5)}R_1 + \frac{2\kappa_5\zeta^{(3)}}{\zeta^{(2)}}\,{}^{(5)}\overset{\star}{T}\,{}^{(5)}\overset{\star}{R}_1 \\ & + \frac{1}{4\zeta^{(2)}}\left[\frac{5}{2\gamma} - \zeta^{(1)}\right]{}^{(5)}R_1 = \frac{1}{4\zeta^{(2)}}\left[\kappa_5\left(\frac{7}{2\gamma} - \zeta^{(1)}\right){}^{(5)}T\right. \\ & \left. + \frac{5}{2}\zeta\right] + \kappa_5\,{}^{(5)}\ddot{T} - \kappa_5\nabla^2\,{}^{(5)}T - \kappa_5\,{}^{(5)}\overset{\star\star}{T}, \end{aligned} \tag{7} $$

where the dot denotes time derivative, the operator ∇^2 is the 3D Laplacian operator, and the star ($\star$) indicates derivative with respect to the fifth extended extra dimension l. It can easily be seen from (7) that given the smallness of γ the dominant contribution in the effective mass term (the coefficient of ${}^{(5)}R_1$) is $8\gamma\zeta^{(2)}$ and hence, as occurs in the usual 4D case, the stability condition in 5D continues to hold, for $f_{,RR} > 0$.

In brane-world scenarios sources of matter in 5D are usually regarded, even in some $f({}^{(5)}R)$ brane-world models [24]. However, in theories like the induced matter approach of Wesson no matter in 5D is considered, so they assume a 5D vacuum. Thus, for these kinds of cases we find that in the absence of matter or in the presence of traceless matter ${}^{(5)}T = 0$, Eq. (5) yields the linearized expression

$$ \begin{aligned} & {}^{(5)}\ddot{R}_1 - {}^{(5)}\overset{\star\star}{R}_1 + \frac{\zeta^{(3)}}{\zeta^{(2)}}\,{}^{(5)}\dot{R}_1^2 - \frac{\zeta^{(3)}}{\zeta^{(2)}}\,{}^{(5)}\overset{\star}{R}{}_1^2 - \nabla^2\,{}^{(5)}R_1 \\ & - \frac{\zeta^{(3)}}{\zeta^{(2)}}(\nabla\,{}^{(5)}R_1)^2 + \frac{1}{4\zeta^{(2)}}\left[\frac{5}{2\gamma} - \zeta^{(1)}\right]{}^{(5)}R_1 = \frac{5\zeta}{8\zeta^{(2)}}. \end{aligned} \tag{8} $$

Again in the effective mass term the dominant contribution comes from $8\gamma\zeta^{(2)}$, and thus the criterion to avoid a negative effective mass remains: $f_{,RR} > 0$, under the presence of traceless 5D matter sources.

2.2 Field equations of both some brane-worlds and induced matter theory as maximally symmetric solutions

Maximally symmetric solutions are very common in $f(R)$ theories of gravity. Due to the Jebsen–Birkhoff theorem in 4D the Schwarzschild solution is no more unique in this kind of theories [25]. In fact, when we go up from 4D to 5D in a theory of the type of general relativity, the Birkhoff theorem is no more valid [11]. Thus, we expect to find more spherically symmetric solutions in a $f(^{(5)}R)$ theory of gravity than on its analog in 4D.

With this idea in mind let us study maximally symmetric solutions in the theory prescribed by the action (1). As is well known, a maximally symmetric solution is characterized by a constant Ricci scalar, in this case by $^{(5)}R = {}^{(5)}R_0$. Hence the trace expression (3) yields

$$f_{,R}(^{(5)}R_0)\,{}^{(5)}R_0 - \frac{5}{2}f(^{(5)}R_0) = \kappa_5\,{}^{(5)}T. \tag{9}$$

The field equation (2) for constant scalar curvature spaces reduces to

$$f_{,R}(^{(5)}R_0)\,{}^{(5)}R_{ab} - \frac{1}{2}f(^{(5)}R_0)g_{ab} = \kappa_5^{(5)}T_{ab}. \tag{10}$$

A combination of (9) and (10) leads to

$$^{(5)}R_{ab} = \frac{\kappa_5{}^{(5)}R_0{}^{(5)}T_{ab} + \frac{1}{2}f(^{(5)}R_0)^{(5)}R_0 g_{ab}}{\kappa_5{}^{(5)}T + \frac{5}{2}f(^{(5)}R_0)}. \tag{11}$$

For traceless 5D matter fields, Eq. (11) leads to

$$^{(5)}R_{ab} = \kappa_{eff5}{}^{(5)}T_{ab} + \frac{1}{5}{}^{(5)}R_0 g_{ab}, \tag{12}$$

where $\kappa_{eff5} = [2\kappa_5{}^{(5)}R_0]/[5f(^{(5)}R_0)]$. These are the field equations of brane-world scenarios with traceless 5D energy-momentum tensor and a 5D cosmological constant term.

In the absence of matter sources $^{(5)}T_{ab} = 0$, Eq. (11) becomes

$$^{(5)}R_{ab} = \frac{1}{5}\,{}^{(5)}R_0 g_{ab}, \tag{13}$$

which for $^{(5)}R_0 = 0$ correspond to the 5D field equations of the induced matter theory of gravity [13]. When $^{(5)}R_0 > 0$ it describes a de Sitter space-time, but when $^{(5)}R_0 < 0$ this space corresponds to an anti-de Sitter space-time, which is the one employed for example in Randall–Sundrum models [9,10].

Another important brane scenario is the known as DGP brane-world [26]. In order to include this setting in our $f(^{(5)}R)$ description, we can start from the 5D action

$$^{(5)}S = \frac{1}{2\kappa_5}\int d^5y\sqrt{g_5}\left[f(^{(5)}R) + \mathcal{L}_m(g_{ab},\psi)\right] + \int d^4x\sqrt{g_4}\left[\frac{1}{2\kappa_5}K^{\pm} + \mathcal{L}_{\text{brane}}\right], \tag{14}$$

where $K^{\pm}$ is the trace of the extrinsic curvature tensor with the $+$ and $-$ denoting different sides of the brane and $\mathcal{L}_{\text{brane}}$ is the 4D lagrangian density defined on the brane [27]. The field equations obtained from (14) read

$$f_{,R}\,{}^{(5)}R_{ab} - \frac{1}{2}fg_{ab} - \left[\nabla_a\nabla_b - g_{ab}\,{}^{(5)}\Box\right]f_{,R} = \kappa_5\left[{}^{(5)}T_{ab} + \delta(l)\tau_{ab}\right], \tag{15}$$

where

$$\tau_{\mu\nu} = -2\frac{\delta(\sqrt{g_4}\mathcal{L}_{\text{brane}})}{\delta g^{\mu\nu}} + g_{\mu\nu}(\sqrt{g_4}\mathcal{L}_{\text{brane}}), \tag{16}$$

is the effective energy-momentum tensor localized on the brane [26,27]. The trace equation derived from (15) is then

$$4\,{}^{(5)}\Box f_{,R} + f_{,R}\,{}^{(5)}R - \frac{5}{2}f = \kappa_5\left[{}^{(5)}T + \delta(l)\tau\right], \tag{17}$$

where $\tau = g^{ab}\tau_{ab}$. A combination of Eqs. (15) and (17) for the case of maximally symmetric solutions i.e. when $^{(5)}R = {}^{(5)}R_0$, leads to

$$^{(5)}R_{ab} = \frac{\kappa_5\,{}^{(5)}R_0\left[{}^{(5)}T_{ab} + \delta(l)\tau_{ab}\right] + \frac{1}{2}\,{}^{(5)}R_0 f(^{(5)}R_0)g_{ab}}{\kappa_5\left({}^{(5)}T + \delta(l)\tau\right) + \frac{5}{2}f(^{(5)}R_0)}. \tag{18}$$

Thus for tracelesss 5D matter fields $^{(5)}T = 0$ and $\tau = 0$, Eq. (18) reduces to

$$^{(5)}R_{ab} = k_{eff5}\left({}^{(5)}T_{ab} + \delta(l)\tau_{ab}\right) + \frac{1}{2}\,{}^{(5)}R_0 g_{ab}, \tag{19}$$

which correspond to the 5D field equations of DGP brane-world scenarios for traceless $^{(5)}T_{ab}$ and τ_{ab}, in the presence of the 5D cosmological constant $\Lambda_5 = (1/2)\,{}^{(5)}R_0$.

In summary, we can say in a fashion that both the field equations of the induced matter theory and the ones of some brane-world models, like for instead some Randall–Sundrum and DGP brane-world models, can be obtained from a $f(^{(5)}R)$ theory of gravity as particular maximally symmetric solutions.

3 The induced 4D field equations

We are now in a position to derive the 4D field equations induced from the 5D dynamics. In order to do so, we will follow the dimensional reduction procedure employed in the Wesson induced matter theory [11–13]. For the sake of generality of the reduction procedure from 5D to 4D, we will consider ${}^{(5)}T_{ab} \neq 0$. However, our main interest here is in the case in which ${}^{(5)}T_{ab} = 0$. A similar consideration has been done in [28].

We use coordinates in which the 5D line element can be written as

$$\mathrm{d}S_5^2 = g_{\alpha\beta}(x^\sigma, l)\mathrm{d}x^\alpha \mathrm{d}x^\beta + \epsilon\Phi^2(x^\sigma, l)\mathrm{d}l^2, \tag{20}$$

where $\epsilon = \pm 1$ accounts for the signature of the extra non-compact coordinate l and $\Phi(x^\sigma, l)$ is a well-behaved metric function.

Now, let us assume that the 5D space-time can be foliated by a family of hypersurfaces, generically defined by $\Sigma_l : l = l(x^\mu)$. Hence, we can consider for example a constant foliation $\Sigma_0 : l = l_0$ or a dynamical one $\Sigma_t : l = l(t)$ [29]. Thus, the 4D line element induced on every hypersurface Σ_l is given by

$$\mathrm{d}S_4^2 = h_{\mu\nu}(x^\sigma)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{21}$$

where $h_{\mu\nu}(x^\sigma) = g_{\mu\nu}(x^\sigma, l)\big|_{\Sigma_l}$ is the 4D induced metric tensor.

Some useful quantities and operators in order to implement the dimensional reduction from 5D to 4D, can be expressed in their (4+1) form as

$$\nabla_\mu \nabla_\nu f_{,R} = \mathcal{D}_\mu \mathcal{D}_\nu f_{,R} + \frac{\epsilon}{2\Phi^2}\overset{\star}{g}_{\mu\nu}\overset{\star}{f}_{,R}, \tag{22}$$

$$\nabla_l \nabla_l f_{,R} = \overset{\star\star}{f}_{,R} + \epsilon\Phi\left(\mathcal{D}_\alpha \Phi\right)\left(\mathcal{D}^\alpha f_{,R}\right) - \frac{\overset{\star}{\Phi}}{\Phi}\overset{\star}{f}_{,R}, \tag{23}$$

$$\begin{aligned} {}^{(5)}\Box f_{,R} = \Box f_{,R} &- \frac{\left(\mathcal{D}_\alpha \Phi\right)\left(\mathcal{D}^\alpha f_{,R}\right)}{\Phi} \\ &+ \frac{\epsilon}{\Phi^2}\left[\overset{\star\star}{f}_{,R} + \left(\frac{1}{2}g^{\mu\nu}\overset{\star}{g}_{\mu\nu} - \frac{\overset{\star}{\Phi}}{\Phi}\right)\overset{\star}{f}_{,R}\right], \end{aligned} \tag{24}$$

$$\begin{aligned} {}^{(5)}R_{\mu\nu} = R_{\mu\nu} - \frac{\mathcal{D}_\mu \mathcal{D}_\nu \Phi}{\Phi} + \frac{\epsilon}{2\Phi^2}\Bigg(&\frac{\overset{\star}{\Phi}}{\Phi}\overset{\star}{g}_{\mu\nu} - \overset{\star\star}{g}_{\mu\nu} \\ &+ g^{\lambda\sigma}\overset{\star}{g}_{\mu\lambda}\overset{\star}{g}_{\sigma\nu} - \frac{1}{2}g^{\alpha\beta}\overset{\star}{g}_{\alpha\beta}\overset{\star}{g}_{\mu\nu}\Bigg), \end{aligned} \tag{25}$$

$$\begin{aligned} {}^{(5)}R_{ll} = -\epsilon\Phi\Box\Phi - \frac{1}{4}\overset{\star}{g}^{\alpha\beta}\overset{\star}{g}_{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}\overset{\star\star}{g}_{\alpha\beta} \\ + \frac{1}{2}\frac{\overset{\star}{\Phi}}{\Phi}g^{\alpha\beta}\overset{\star}{g}_{\alpha\beta}, \end{aligned} \tag{26}$$

where $\mathcal{D}_\mu$ denotes the 4D covariant derivative and $\Box = h^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu$ is the 4D D'Alambertian operator. Thus, with the help of (22) to (26) the $\mu\nu$ and ll components of the field equation (2) can be combined to obtain on 4D hypersurfaces

$$\begin{aligned} &\left[f_{,R}\, R_{\mu\nu} - \frac{1}{2}f g_{\mu\nu} - \left(\mathcal{D}_\mu\mathcal{D}_\nu - g_{\mu\nu}\Box\right)f_{,R}\right]\Bigg|_{\Sigma_l} \\ &\quad = \kappa_5\, {}^{(5)}T_{\mu\nu}\Big|_{\Sigma_l} + \kappa_4 f_{,R}\tau_{\mu\nu}, \end{aligned} \tag{27}$$

where the matter tensor $\tau_{\mu\nu}$ is defined by [11–13]

$$\begin{aligned} \kappa_4\tau_{\mu\nu} = \kappa_4 T^{(IM)}_{\mu\nu} &+ \frac{\epsilon}{2\Phi^2}\frac{\overset{\star}{g}_{\mu\nu}\overset{\star}{f}_{,R}}{f_{,R}} \\ &- g_{\mu\nu}\frac{\left(\mathcal{D}_\alpha\Phi\right)\left(\mathcal{D}^\alpha f_{,R}\right)}{\Phi f_{,R}} \\ &- \frac{\epsilon\, g_{\mu\nu}}{\Phi^2}\left[\frac{\overset{\star\star}{f}_{,R}}{f_{,R}} + \left(\frac{1}{2}g^{\alpha\beta}\overset{\star}{g}_{\alpha\beta} - \frac{\overset{\star}{\Phi}}{\Phi}\right)\frac{\overset{\star}{f}_{,R}}{f_{,R}}\right] \\ &- \frac{1}{4}g_{\mu\nu}\left[\overset{\star}{g}^{\lambda\sigma}\overset{\star}{g}_{\lambda\sigma} + \left(g^{\lambda\sigma}\overset{\star}{g}_{\lambda\sigma}\right)^2\right] \end{aligned} \tag{28}$$

$T^{(IM)}_{\alpha\beta}$ being the energy-momentum tensor for geometrically induced matter, which was first introduced in the Wesson induced matter theory [12], which is given by

$$\begin{aligned} \kappa_4 T^{(IM)}_{\mu\nu} = \frac{h_{\mu\nu}\Box\Phi}{\Phi} - \frac{\epsilon}{2\Phi^2}\Bigg\{&\frac{\overset{\star}{\Phi}}{\Phi}\overset{\star}{h}_{\mu\nu} - \overset{\star\star}{h}_{\mu\nu} + h^{\lambda\alpha}\overset{\star}{h}_{\mu\lambda}\overset{\star}{h}_{\nu\alpha} \\ &- \frac{1}{2}h^{\alpha\beta}\overset{\star}{h}_{\alpha\beta}\overset{\star}{h}_{\mu\nu} + \frac{1}{4}h_{\mu\nu}\left[\overset{\star}{h}^{\alpha\beta}\overset{\star}{h}_{\alpha\beta} + (h^{\alpha\beta}\overset{\star}{h}_{\alpha\beta})^2\right]\Bigg\}. \end{aligned} \tag{29}$$

In order to evaluate the $f({}^{(5)}R)$ terms in (27) on the 4D hypersurface Σ_l, we express the 5D Ricci scalar curvature as a function of its analog 4D in the form [11–13]

$$\begin{aligned} {}^{(5)}R = {}^{(4)}R - \frac{\Box\Phi}{\Phi} + \frac{\epsilon}{2\Phi^2}\Bigg(&\frac{\overset{\star}{\Phi}}{\Phi}g^{\mu\nu}\overset{\star}{g}_{\mu\nu} - g^{\mu\nu}\overset{\star\star}{g}_{\mu\nu} \\ &+ g^{\mu\nu}g^{\lambda\alpha}\overset{\star}{g}_{\mu\lambda}\overset{\star}{g}_{\alpha\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\overset{\star}{g}_{\alpha\beta}\overset{\star}{g}_{\mu\nu}\Bigg), \end{aligned} \tag{30}$$

which can also be written in terms of the extrinsic curvature tensor $K_{\alpha\beta}$ as ${}^{(5)}R = {}^{(4)}R - (K^{\mu\nu}K_{\mu\nu} - K^2)$, with $K = h^{\alpha\beta}k_{\alpha\beta}$. Thus, using (30) the field equation (27) on the 4D space-time Σ_l read

$$\begin{aligned} &F'({}^{(4)}R, \varphi)R_{\mu\nu} - \frac{1}{2}F({}^{(4)}R, \varphi)h_{\mu\nu} \\ &\quad - \left[\mathcal{D}_\mu\mathcal{D}_\nu - h_{\mu\nu}\Box\right]F'({}^{(4)}R, \varphi) \\ &\quad = \kappa_4 S_{\mu\nu} + \kappa_4 F'({}^{(4)}R, \varphi)\tau_{\mu\nu}, \end{aligned} \tag{31}$$

where the prime denotes a derivative with respect to ${}^{(4)}R$, the function $F({}^{(4)}R, \varphi) = f[{}^{(5)}R = {}^{(4)}R + E]|_{\Sigma_l}$ is the induced function of the 4D Ricci scalar, $\kappa_4 S_{\mu\nu} = [\kappa_5(T_{\mu\nu} - g_{\mu\nu}({}^{(5)}T_{ll} - (\epsilon\Phi^2/3)\,{}^{(5)}T))]|_{\Sigma_l}$ and $\varphi(x^\sigma) = E(x^\sigma, l)|_{\Sigma_l}$, the extrinsic scalar curvature parameter E being defined by

$$\begin{aligned} E(x^a) = -\frac{\Box\Phi}{\Phi} + \frac{\epsilon}{2\Phi^2}\Bigg(&\frac{\overset{\star}{\Phi}}{\Phi}g^{\mu\nu}\overset{\star}{g}_{\mu\nu} - g^{\mu\nu}\overset{\star\star}{g}_{\mu\nu} \\ &+ g^{\mu\nu}g^{\lambda\alpha}\overset{\star}{g}_{\mu\lambda}\overset{\star}{g}_{\alpha\nu} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\overset{\star}{g}_{\alpha\beta}\overset{\star}{g}_{\mu\nu}\Bigg). \end{aligned} \tag{32}$$

The field equation (31) describe a new type of $F({}^{(4)}R,\varphi)$ theory where a matter coupling of the form $F'\tau_{\mu\nu}$ is present. When the parameter $E(x^a)$ depends only of the extra coordinate, the effective scalar field φ becomes a constant, and in this case the $F({}^{(4)}R,\varphi)$ transforms into a $F({}^{(4)}R)$ theory with the same matter coupling. As we will see in a forthcoming example, when the parameter E has only time dependence, the $F({}^{(4)}R,\varphi)$ resulting theory can be interpreted as an effective modified gravity theory with dynamical coefficients. If a 5D vacuum is considered, then the sources of matter are exclusively induced by the 5D geometry of the theory. Notice that the modified gravity theory described by Eq. (31) is in fact an effective theory, in the sense that it is derived from the 5D one described by the action (1).

For example, for the line element

$$\mathrm{d}s_5^2=\left(\frac{l}{l_0}\right)^{2\alpha}\left[\mathrm{d}t^2-a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]-\Phi(t)^2\mathrm{d}l^2, \qquad (33)$$

the extrinsic scalar curvature parameter $E(x^a)$ reads

$$E(t)=2\left[\frac{\ddot{\Phi}}{\Phi}+3H\frac{\dot{\Phi}}{\Phi}-\frac{2\alpha(5\alpha-2)}{l^2\Phi^2}\right]. \qquad (34)$$

Thus, on our 4D space-time $\Sigma_0\,:\,l=l_0$ the effective scalar field is given by

$$\varphi(t)=2\left[\frac{\ddot{\Phi}}{\Phi}+3H\frac{\dot{\Phi}}{\Phi}-\frac{2\alpha(5\alpha-2)}{l_0^2\Phi^2}\right]. \qquad (35)$$

Hence, if we consider for example a $f({}^{(5)}R)={}^{(5)}R+\frac{b(t)}{{}^{(5)}R}$, in 4D the induced effective $F({}^{(4)}R,\varphi)$ has the form

$$F({}^{(4)}R,\varphi)={}^{(5)}R+\varphi(t)+\frac{b(t)}{{}^{(4)}R+\varphi(t)}. \qquad (36)$$

This $F({}^{(4)}R,\varphi)$ can be interpreted as a $F({}^{(4)}R)$ modified gravity theory with dynamical coefficients, similar to the one proposed in [14]. As it was shown in [14], this kind of models are very useful to describe the dark energy dominance epoch in the evolution of the universe. A remarkable difference between the models in [14] and the example given by (36) relies in the form of the field equations. Moreover, in here the scalar field φ is not a dynamical coefficient introduced apriori, instead it is determined by the 5D geometry.

Now, let us to give another application. If we consider for example a 5D line element of a warped product space-time

$$\mathrm{d}s_5^2=e^{2A(l)}\left[\mathrm{d}t^2-a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]-\mathrm{d}l^2, \qquad (37)$$

in this case the extrinsic scalar curvature parameter $E(x^a)$ results

$$E(l)=-4\left(2\overset{\star\star}{A}+5\overset{\star}{A}{}^2\right). \qquad (38)$$

Evaluating (38) on the 4D hypersurface $\Sigma_0\,:\,l=l_0$, the effective scalar field has the expression

$$\varphi(l_0)=-\,4\left[2\overset{\star\star}{A}+5\overset{\star}{A}{}^2\right]\Big|_{l=l_0}, \qquad (39)$$

which clearly is a constant. Therefore, the effective $F({}^{(4)}R,\varphi)$ effective theory becomes a matter coupled $F({}^{(4)}R)$ modified gravity theory in this case.

In the both examples given previously, when no matter sources are considered in 5D, i.e. when ${}^{(5)}T_{ab}=0$, the matter in 4D is induced by the 5D geometry and it is described by the 4D energy-momentum tensor $\tau_{\mu\nu}$ whose formula is given in (28). This energy-momentum tensor is the analogous of the energy-momentum tensor that appears in the Wesson induced matter theory.

Now, returning to the non-diagonal components of the field equation (2), the components μl of the field Eq. (2) can be written as

$$(\Phi f_{,R})\mathcal{D}_\alpha\mathcal{P}^\alpha_\mu=\kappa_5{}^{(5)}T_{\mu l}+\frac{1}{2}g^{\alpha\sigma}\overset{\star}{g}_{\sigma\mu}f_{,RR}R_{,\alpha}-\frac{\Phi_{,\mu}}{\Phi}\overset{\star}{f}_{,R}$$
$$+\overset{\star}{f}_{,RR}R_{,\mu} \qquad (40)$$

where

$$\mathcal{P}_{\alpha\beta}=\frac{1}{2\Phi}\left(\overset{\star}{g}_{\alpha\beta}-g_{\alpha\beta}g^{\mu\nu}\overset{\star}{g}_{\mu\nu}\right). \qquad (41)$$

A similar equation to (40) is obtained in the induced matter theory of gravity [13]. In that theory the analogous expression is

$$\mathcal{D}_\alpha\mathcal{P}^\alpha_\mu=0. \qquad (42)$$

The conservation like Eq. (42) can be recovered from (40) when we consider $f({}^{(5)}R)={}^{(5)}R$ in vacuum i.e. without any sources of matter in 5D.

3.1 Dolgov–Kawasaki instability criterion for the effective $F({}^{(4)}R,\varphi)$ theory

In order to study the Dolgov–Kawasaki instability in the matter sector of the effective $F({}^{(4)}R,\varphi)$ theory induced from a $f({}^{(5)}R)$ gravity theory, we will proceed as follows.

The trace of the field equation (31) leads to

$$3\Box F'+F'\,{}^{(4)}R-2F=\kappa_4 S+\kappa_4 F'\tau, \qquad (43)$$

where $S=h^{\mu\nu}S_{\mu\nu}$ and $\tau=h^{\mu\nu}\tau_{\mu\nu}$. Now, deviations from Einstein's general relativity of our $F({}^{(4)}R,\varphi)$ are described by the expression

$$F({}^{(4)}R,\varphi)={}^{(4)}R+\sigma Z({}^{(4)}R,\varphi), \qquad (44)$$

where σ is a small parameter with $[\text{length}]^{-2}$ units. Employing (44) the trace equation (43) yields

$$\Box\,{}^{(4)}R + \frac{Z'''}{Z'}\mathcal{D}^{\mu}\,{}^{(4)}R\mathcal{D}_{\mu}\,{}^{(4)}R + \frac{1}{Z''}\frac{\partial^2 Z'}{\partial\varphi^2}\mathcal{D}^{\mu}\varphi\mathcal{D}_{\mu}\varphi + \frac{3\sigma}{Z''}\frac{\partial Z'}{\partial\varphi}\Box\varphi + \frac{(\sigma Z'-2)\,{}^{(4)}R}{3\sigma Z''} = \kappa_4\frac{(S+\sigma Z'\tau)}{3\sigma Z''} + \frac{2Z}{3Z''}. \tag{45}$$

In the weak field regime we can use the approximation

$$h_{\alpha\beta} = \eta_{\alpha\beta} + \gamma_{\alpha\beta}, \quad {}^{(4)}R = R_b + R_1, \tag{46}$$

where $\eta_{\alpha\beta}$ is the 4D Minkowsky metric, $\gamma_{\alpha\beta}$ is a fluctuation of the metric with respect to the Minkowsky background, $|R_1/R_b| \ll 1$ and $R_b = -\kappa_4(S + F'\tau)$. Using (46), Eq. (45), to first order in R_1, reads

$$\begin{aligned}
&\ddot{R}_1 - \nabla^2 R_1 - \frac{2\kappa_4 Z'''}{Z''}\dot{S}\dot{R}_1 - \frac{2\kappa_4 Z'''}{Z''}\left(\dot{Z}'\tau + Z'\dot{\tau}\right)\dot{R}_1 \\
&\quad + \frac{2\kappa_4 Z'''}{Z''}\nabla S\cdot\nabla R_1 + \frac{2\kappa_4 Z'''}{Z''}\left(\nabla Z'\tau + Z'\nabla\tau\right)\cdot\nabla R_1 \\
&\quad + \frac{1}{3Z''}\left(\frac{1}{\sigma} - Z'\right)R_1 + \frac{1}{Z''}\frac{\partial^2 Z'}{\partial\varphi^2}\mathcal{D}^{\mu}\varphi\mathcal{D}_{\mu}\varphi + \frac{3\sigma}{Z''}\frac{\partial Z'}{\partial\varphi}\Box\varphi \\
&\quad = \kappa_4\ddot{S} + \kappa_4\sigma\left(\ddot{Z}'\tau + 2\dot{Z}'\dot{\tau} + Z'\ddot{\tau}\right) \\
&\qquad - \kappa_4\left[\nabla^2 S + \nabla^2(\sigma Z'\tau)\right] - \frac{\kappa_4\left(S + \sigma Z'\tau\right)Z'}{3Z''} - \frac{Z}{3Z''}.
\end{aligned} \tag{47}$$

Clearly the effective mass term is dominated by the factor $3\sigma Z''$ and then the theory is stable only if $F'' > 0$.

4 Final comments

In this letter, using some ideas of the Wesson induced matter theory of gravity [11–13], we have discussed some implications of a 5D gravity governed by a $f({}^{(5)}R)$ theory on spacetimes with a non-compact spacelike fifth extra coordinate.

One interesting result is that the 5D field equations of some brane-world theories and the ones of the induced matter theory, can be obtain as maximally symmetric solutions of the same $f({}^{(5)}R)$ theory, under certain conditions imposed on 5D energy-momentum tensor.

We are interested in 5D vacuum solutions and hence on our theoretical setting the dimensional reduction mechanism adopted is the one employed in the induced matter theory [11–13]. Taking a foliation of the 5D space-time on the fifth extra coordinate, our 4D universe is described by a generic hypersurface Σ_l, embedded into the 5D space-time. On Σ_l we obtain a set of induced 4D field equations corresponding to a $F({}^{(4)}R, \varphi)$ modified gravity theory, which exhibits a matter coupling term of the form $\kappa_4 F'({}^{(4)}R, \varphi)\tau_{\mu\nu}$. This coupling is different from the one in BBHL theory [21]. If we consider a 5D vacuum (${}^{(5)}T_{ab} = 0$), matter sources in our 4D universe are induced geometrically by the 5D geometry in a similar manner as it is done in the Wesson induced matter theory. However, if matter sources are regarded, it would be necessary to impose restrictions on the 5D energy-momentum tensor ${}^{(5)}T_{ab}$ for consistency of the theory. Unfortunately, these last cases go out of the scope of this paper.

In general in a BBHL theory [21], in order to recover the Einstein–Hilbert action it is necessary to specify the two functions of the scalar curvature as $f_1(R) = R$ and $f_2(R) = 1$. In our case it is sufficient to fix $f({}^{(5)}R) = {}^{(5)}R$ and take for example a warped product metric, and automatically $F({}^{(4)}R, \varphi) = {}^{(4)}R + \varphi_0$ and $F'({}^{(4)}R, \varphi) = 1$, hence resulting that the field Eq. (31) become the ones of general relativity. Thus the matter coupling term is governed by the same $F({}^{(4)}R, \varphi)$, instead to fix two different functions as in the case of BBHL theory. The Dolgov–Kawasaki instability criterion of both the $f({}^{(5)}R)$ theory and the effective $F({}^{(4)}R, \varphi)$ continues being the same as in the usual $f(R)$ theories: $f_{,RR}({}^{(5)}R) > 0$ and $F''({}^{(4)}R, \varphi) > 0$.

Even when the effective field equation (31) are not the ones of general relativity, it is possible to force them to have that form by means of the introduction of an effective energy-momentum tensor. Therefore, as happens in the usual $f(R)$ theories, this redefined energy-momentum tensor will be non-positive definite and none of the energy conditions will hold. However, as it is done in conventional $f(R)$ theories, it is possible to impose those energy conditions for the effective energy-momentum tensor. This restricts some physical parameters of the model considered. However, there are some cosmological applications of $f(R)$-gravity in which the violation of the energy conditions makes it possible to have $\dot{H} > 0$ and bouncing universes [4,30].

When the effective scalar field φ becomes only time dependent, the resulting $F({}^{(4)}R, \varphi)$ theory can be interpreted as a modified gravity theory with dynamical coefficients. However, one difference between the effective theory resulting from our formalism and the ones in the literature relies on their field equations [which in our case are given by (31)]. An interesting point is that according to [14], modified gravity models with dynamical coefficients, may be viable to explain the present accelerated expansion of the universe. The study of cosmological solutions of the effective $F({}^{(4)}R, \varphi)$ theory we have obtained will be matter of future work.

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References

1. S. Permutter et al., Nature **391**, 51 (1998)
2. J. Wang, Y. Wu, Y. Guo, F. Pi, Y. Zhao, X. Sun, Eur. Phys. J. C **69**, 541–546 (2010)
3. C. Brans, R.H. Dicke, Phys. Rev. **124**, 124 (1961)
4. V. Faraoni, *Cosmology in scalar-tensor gravity* (Kluwer Academic, Dordrecht, 2004)
5. S. Capozziello, V. Faraoni, *Beyond einstein gravity* (Springer, Dordrecht, 2011)
6. S. Nojiri, S.D. Odintsov, Phys. Rep. **505**, 59–144 (2011)
7. Dvali et al., Phys. Lett. B **485**, 208 (2000)
8. R. Maartens, Living Rev. Relat. **7**, 7 (2004)
9. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)
10. L. Randall, R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999)
11. P.S. Wesson, Space-time-matter, World Scientific Singapore (1999)
12. J.M. Overduin, P.S. Wesson, Phys. Rep. **283**, 302 (1997)
13. P.S. Wesson, Five-dimensional physics, World Scientific Singapore (2006)
14. S. Nojiri, S.D. Odintsov, Phys. Lett. B **599**, 137–142 (2004)
15. T.P. Sotiriou, Phys. Lett. B **664**, 225–228 (2008)
16. A.D. Dolgov, M. Kawasaki, Phys. Lett. B **573**, 1 (2003)
17. V. Faraoni, Phys. Rev. D **74**, 104017 (2006). arXiv:astro-ph/0610734
18. S. Nojiri, S.D. Odintsov, Phys. Rev D **68**, 123512 (2003)
19. I. Navarro, K. Acoleyen, J. Cosmol. Astropart. Phys. **0603**, 008 (2006). arXiv:gr-qc/0512109
20. G. Allemandi, A. Borowiec, M. Francaviglia, S.D. Odintsov, Phys. Rev. D **72**, 063505 (2005)
21. O. Bertolami, C.G. Bohmer, T. Harko, F.S.N. Lobo, Phys. Rev. D **75**, 104016 (2007)
22. V. Faraoni, Phys. Rev. D **76**, 127501 (2007)
23. M. Milgrom, New Astron. Rev. **46**, 741 (2002)
24. D. Bazeia, A.S. Lobao Jr., R. Menezes, A.Yu. Petrov, A.J. da Silva, Phys. Lett. B **729**, 127–135 (2014). arXiv:1311.6294 [gr-qc]
25. T.P. Sotirion, V. Faraoni, Rev. Mod. Phys. **82**, 451–497 (2010)
26. G. Dvali, G. Gabadadze, M. Porrati, Phys. Lett. B **485**, 208–214 (2000)
27. J. Saavedra, Y. Vásquez, JCAP **0904**, 013 (2009)
28. J. Ponce de Leon, JCAP **1003**, 030 (2010)
29. J. Ponce de Leon, Mod. Phys. Lett. A **21**, 947–959 (2006)
30. T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451–497 (2010)